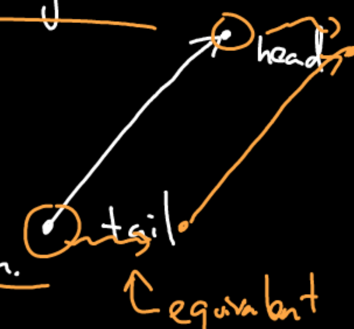
Vectors: Algebra for Geometry

Goal: Develop an algebraic framework for multivariate calculus.

Defⁿ: A vector is a directed line segment.

Vectors have a head and a tail.

Vectors are equivalent when they can be obtained from one another via translation.

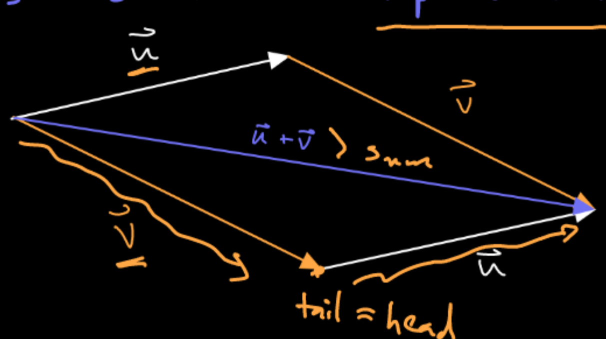


NB: Usually we think of vectors as having their tails at the origin. In this way, we can identify each vector with its head. In $\mathbb{R}^3$, we denote the vector with head (x, y, z) by $\langle x, y, z \rangle$. It is important to remember that the head is NOT the vector, but a representation of it which we can use for calculation...

Vector Operations

Vector addition is accomplished via the parallelogram law.

This is the "tip-to-tail" addition...



Algebraically, if

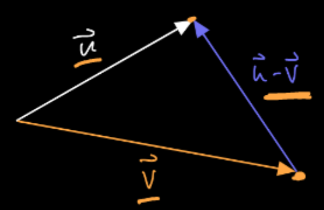
$\vec{u} = \langle u_1, u_2, u_3 \rangle$ and

$\vec{v} = \langle v_1, v_2, v_3 \rangle$ then

$$\vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$$

Vector subtraction is accomplished via the **triangle law**.

This is a "tip-to-tip" operation.



Algebraically, if $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\vec{u} - \vec{v} = \langle u_1, u_2, u_3 \rangle - \langle v_1, v_2, v_3 \rangle = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

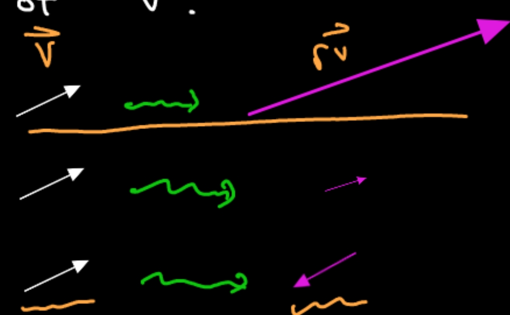
Scalar multiplication is accomplished by scaling and (possibly) redirecting a given vector.

Given real number (i.e. scalar) $r \in \mathbb{R}$ and vector $\vec{v}$, vector $r\vec{v}$ is a scaled version of $\vec{v}$.

If $|r| \geq 1$, $\vec{v}$ is stretched out.

If $0 \leq |r| \leq 1$, $\vec{v}$ is compacted.

If $r < 0$, $\vec{v}$ is reversed.



Thus scalar multiplication acts on both the direction and the magnitude of the vector in question.
↳ length

Algebraically, if $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then

$$r\vec{v} = r\langle v_1, v_2, v_3 \rangle = \langle rv_1, rv_2, rv_3 \rangle.$$

Properties of Vector Operations

NB: Vector operations all act on the components of the vectors involved. Thus we can use them to understand the algebra of vectors :).

Prop: If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ are vectors and $a, b \in \mathbb{R}$ are scalars, then...

① (commutativity) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ ←

$\vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
$= \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle = \langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \vec{v} + \vec{u}$

② (associativity) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ ←

$\vec{u} + (\vec{v} + \vec{w}) = \langle u_1, u_2, u_3 \rangle + \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle = \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3) \rangle$
$= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle + \vec{w}$

③ (zero vector) $\vec{u} + \vec{0} = \vec{u}$ ←

$\vec{u} + \vec{0} = \langle u_1, u_2, u_3 \rangle + \langle 0, 0, 0 \rangle = \langle u_1 + 0, u_2 + 0, u_3 + 0 \rangle = \vec{u}$

④ (inverses) $\vec{u} + (-\vec{u}) = \vec{0}$ ←

$\vec{u} + (-\vec{u}) = \langle u_1, u_2, u_3 \rangle + \langle -u_1, -u_2, -u_3 \rangle = \langle u_1 - u_1, u_2 - u_2, u_3 - u_3 \rangle = \vec{0}$

⑤ (Scalar distribution) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ ←

$a(\vec{u} + \vec{v}) = a\langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle = \langle a(u_1 + v_1), a(u_2 + v_2), a(u_3 + v_3) \rangle$
$= \langle au_1 + av_1, au_2 + av_2, au_3 + av_3 \rangle = a\langle u_1, u_2, u_3 \rangle + a\langle v_1, v_2, v_3 \rangle$

⑥ (Scalar action) $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ ←

$(a + b)\langle u_1, u_2, u_3 \rangle = \langle (a+b)u_1, (a+b)u_2, (a+b)u_3 \rangle = \langle au_1 + bu_1, au_2 + bu_2, au_3 + bu_3 \rangle$
$= a\langle u_1, u_2, u_3 \rangle + b\langle u_1, u_2, u_3 \rangle$

⑦ (Scalar association) $a(b\vec{u}) = (ab)\vec{u}$ ←

$a\langle bu_1, bu_2, bu_3 \rangle = \langle a(bu_1), a(bu_2), a(bu_3) \rangle = \langle (ab)u_1, (ab)u_2, (ab)u_3 \rangle$

⑧ (identity) $1\vec{u} = \vec{u}$ and $0\vec{u} = \vec{0}$

NB: I've written these properties about $\mathbb{R}^3$, but the same reasoning yields the exact properties in $\mathbb{R}^n$.

Indeed, we've shown that $\mathbb{R}^3$ (and more generally $\mathbb{R}^n$) is a real vector space...

Recall that vectors have magnitude and direction. We can express these via components as well!! :)

Prop: The magnitude of vector $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$ is precisely $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. ←

"length of $\vec{v}$" ↗

↑ hey, that's just the distance formula!

We thus obtain the following fundamental fact:

Prop: If $r \in \mathbb{R}$ and $v \in \mathbb{R}^3$, then $|r\vec{v}| = |r||\vec{v}|$.

pf: $|r\vec{v}| = |\langle rv_1, rv_2, rv_3 \rangle| = \sqrt{(rv_1)^2 + (rv_2)^2 + (rv_3)^2}$

$= \sqrt{r^2(v_1^2 + v_2^2 + v_3^2)} = \sqrt{r^2}\sqrt{v_1^2 + v_2^2 + v_3^2} = |r||\vec{v}|$ ☒

The direction of a nonzero vector is encoded by its corresponding unit vector, i.e. that vector scaled to have length 1. The unit vector in the direction of $\vec{v}$ is the vector $\frac{1}{|\vec{v}|}\vec{v}$.

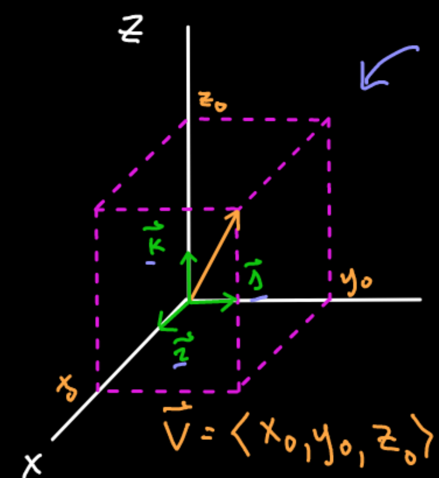
Ex: The unit vector in the direction of $\vec{v} = \langle 1, -1, 3 \rangle$ is: $\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{1^2 + (-1)^2 + 3^2}} \langle 1, -1, 3 \rangle = \left\langle \frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right\rangle$ ☒

Note that there are three special vectors which describe the components of all other vectors in $\mathbb{R}^3$. The standard basis is formed by vectors

$$\vec{i} = \langle 1, 0, 0 \rangle, \quad \vec{j} = \langle 0, 1, 0 \rangle, \text{ and } \vec{k} = \langle 0, 0, 1 \rangle.$$

Using these vectors, we can describe all other vectors!

Indeed, for a vector $\vec{v} = \langle x_0, y_0, z_0 \rangle$, we can express our vector by



$$\vec{v} = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$$

Note that this depends on the fact that we're working in $\mathbb{R}^3$. We can do something similar in $\mathbb{R}^n$, but we'll need more "standard" vectors...

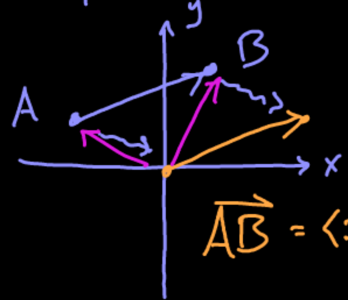
These standard vectors are very important, and will play a big role later on...

EXAMPLES

Ex: Find a vector with representation given by the directed line segment from $A = (-2,1)$ to $B = (1,2)$.

Sol:

picture



$\vec{AB} = \langle 3,1 \rangle$

Algebraically

$$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= \langle 1,2 \rangle - \langle -2,1 \rangle$$
$$= \langle 1-(-2), 2-1 \rangle$$
$$= \langle 3,1 \rangle$$

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Ex: Compute $\vec{u}+\vec{v}$, $4\vec{u}+2\vec{v}$, $|\vec{u}|$, and $|\vec{u}-\vec{v}|$ for

ⓐ vectors $\vec{u} = \langle -3,4 \rangle$ and $\vec{v} = \langle 9,-1 \rangle$

$$\vec{u}+\vec{v} = \langle -3,4 \rangle + \langle 9,-1 \rangle = \langle -3+9, 4-1 \rangle = \langle 6,3 \rangle$$
$$4\vec{u}+2\vec{v} = 4\langle -3,4 \rangle + 2\langle 9,-1 \rangle = \langle -12+18, 16-2 \rangle = \langle 6,14 \rangle$$
$$|\vec{u}| = \sqrt{(-3)^2+4^2} = \sqrt{9+16} = 5$$
$$|\vec{u}-\vec{v}| = |\langle -12,5 \rangle| = \sqrt{(-12)^2+5^2} = \sqrt{169} = 13$$

ⓑ vectors $\vec{u} = 4\vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{v} = 2\vec{i} - 4\vec{k}$.

$\vec{u} = \langle 4,-3,2 \rangle$ $\qquad$ $\vec{v} = \langle 2,0,-4 \rangle$

$$\vec{u}+\vec{v} = \langle 4,-3,2 \rangle + \langle 2,0,-4 \rangle = \langle 6,-3,-2 \rangle$$
$$4\vec{u}+2\vec{v} = 4\langle 4,-3,2 \rangle + 2\langle 2,0,-4 \rangle = \langle 16+4, -12+0, 8-8 \rangle = \langle 20,-12,0 \rangle$$
$$|\vec{u}| = \sqrt{4^2+(-3)^2+2^2} = \sqrt{16+9+4} = \sqrt{29}$$
$$|\vec{u}-\vec{v}| = \sqrt{(4-2)^2+(-3-0)^2+(2-(-4))^2} = \sqrt{2^2+3^2+6^2} = \sqrt{4+9+36} = 7.$$

Ex: Compute the unit vector in the direction of...

(a) $\vec{v} = \langle 6, -2 \rangle$

$$\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{6^2+(-2)^2}}\langle 6,-2\rangle = \frac{1}{2\sqrt{10}}\langle 6,-2\rangle = \left\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle$$

(b) $\vec{v} = 8\vec{i} - \vec{j} + 4\vec{k} = \langle 8,-1,4\rangle$

$$\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{8^2+(-1)^2+4^2}}\langle 8,-1,4\rangle = \frac{1}{\sqrt{81}}\langle 8,-1,4\rangle$$

$$= \left\langle \frac{8}{9}, -\frac{1}{9}, \frac{4}{9} \right\rangle \quad \boxtimes$$

Ex: Compute the vector of length 4 in the direction of vector $\vec{v} = \langle 6,-3,2\rangle$.

Sol: The desired vector is $\quad$ $\{\, |r\vec{u}| = |r|\,|\vec{u}|$

$$4\frac{1}{|\vec{v}|}\vec{v} = \frac{4}{\sqrt{6^2+(-3)^2+2^2}}\langle 6,-3,2\rangle$$

$$= \frac{4}{\sqrt{49}}\langle 6,-3,2\rangle$$

$$= \frac{4}{7}\langle 6,-3,2\rangle$$

$$= \left\langle \frac{24}{7}, -\frac{12}{7}, \frac{8}{7} \right\rangle \quad \boxtimes$$

Ex: What is the angle between the positive x-axis and the vector...

$\uparrow \vec{i} = \vec{u}$

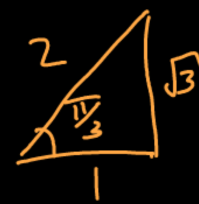
$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$

$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \right) \leftarrow$

ⓐ $\vec{i} + \sqrt{3}\,\vec{j}$

$\theta = \arccos\left(\frac{\vec{i} \cdot (\vec{i} + \sqrt{3}\,\vec{j})}{|\vec{i}|\,|\vec{i} + \sqrt{3}\,\vec{j}|} \right)$

$= \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$



ⓑ $8\vec{i} + 6\vec{j}$

$\theta = \arccos\left(\frac{\vec{i} \cdot (8\vec{i} + 6\vec{j})}{|\vec{i}|\,|8\vec{i} + 6\vec{j}|} \right) = \arccos\left(\frac{8}{1 \cdot \sqrt{8^2 + 6^2}} \right)$

$= \arccos\left(\frac{8}{10} \right) = \arccos\left(\frac{4}{5} \right)$ ▨

Ex: Describe the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $|\vec{r} - \vec{r}_0| = 1$ where $\vec{r} = \langle x, y, z \rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$.

Sol: $1 = |\vec{r} - \vec{r}_0| \iff 1 = |\langle x - x_0, y - y_0, z - z_0 \rangle|$

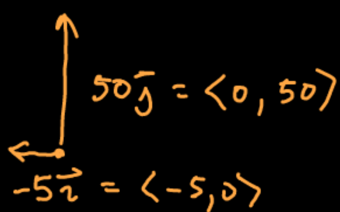
$\iff 1 = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$

$\iff 1^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$

$\therefore$ The shape of this set is a sphere of radius 1 centered at (x_0, y_0, z_0). ▨

Ex: A captain walks due west at a rate of 5 km/h on the deck of a ship moving north at a rate of 50 km/h. Find the speed and direction of the captain relative to the surface of the water.

Sol: picture



$\therefore$ The captain (rel. the surface of the water) has velocity described by the vector $\vec{v} = -5\vec{i} + 50\vec{j} = \langle -5, 50 \rangle$.

$\therefore$ the captain's speed is $|\vec{v}| = \sqrt{(-5)^2 + 50^2} = 5\sqrt{101}$, and she travels in the direction $\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{5\sqrt{101}}\langle -5, 50 \rangle = \langle -\frac{1}{\sqrt{101}}, \frac{10}{\sqrt{101}} \rangle$.